

## ECS315 2016/1 Part IV.2 Dr.Prapun

### 10.4 Families of Continuous Random Variables

Theorem 10.32 states that any nonnegative function  $f(x)$  whose integral over the interval  $(-\infty, +\infty)$  equals 1 can be regarded as a probability density function of a random variable. In real-world applications, however, special mathematical forms naturally show up. In this section, we introduce a couple families of continuous random variables that frequently appear in practical applications. The probability densities of the members of each family all have the same mathematical form but differ only in one or more parameters.

#### 10.4.1 (Continuous) Uniform Distribution

$[a, b)$   
 $(a, b]$   
 $(a, b)$

#### Definition 10.42.

- The (continuous) **uniform** distribution on an interval  $[a, b]$ , is denoted by  $\text{uniform}([a, b])$  or  $\mathcal{U}([a, b])$  or simply  $\mathcal{U}(a, b)$ .

Ex In MATLAB,  $\text{rand} \sim \mathcal{U}(0,1)$

- Expressions that are synonymous with “ $X$  is a uniform random variable” are
  - “ $X$  is uniformly distributed”,
  - “ $X$  has a uniform distribution”,
  - and “ $X$  has a uniform density”.

To specify the support (range) of  $X$ , we may also append “on/over the interval  $(a, b)$ ”.

- This family is characterized by pdf of the form

$$f_X(x) = \begin{cases} 0, & x < a, x > b \\ \frac{1}{b-a}, & a \leq x \leq b \end{cases}$$

- The constants  $a$  and  $b$  are referred to as the parameters of the uniform distribution.

**10.43.** Important Interpretation: A continuous uniform random variable  $X$  on the interval  $[a, b]$  is just as likely to be “near” any value in  $[a, b]$  as any other value.

**10.44.** In MATLAB,

- use `X = a+(b-a)*rand` or `X = random('Uniform',a,b)` to generate  $X \sim \mathcal{U}(a, b)$ ,
- use `pdf('Uniform',x,a,b)` and `cdf('Uniform',x,a,b)` to evaluate the pdf and cdf at  $x$ , respectively.

**Exercise 10.45.** Show that  $F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x > b. \end{cases}$

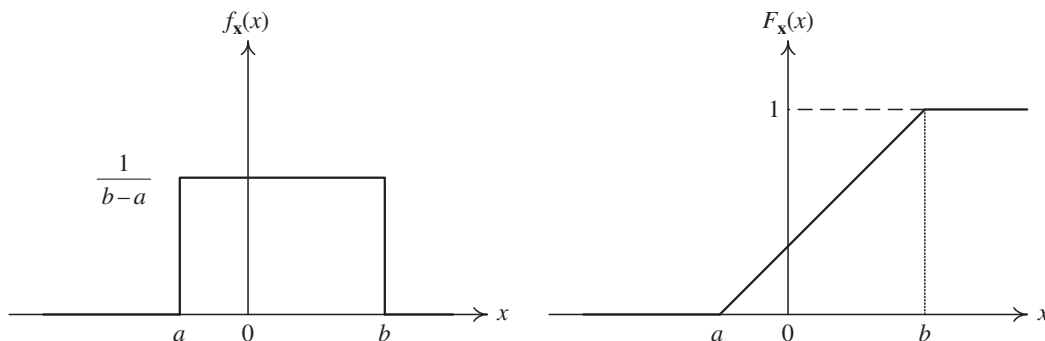
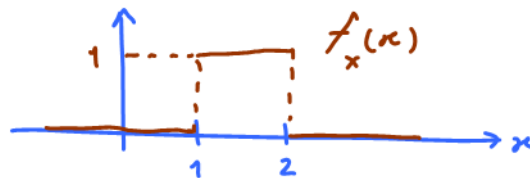


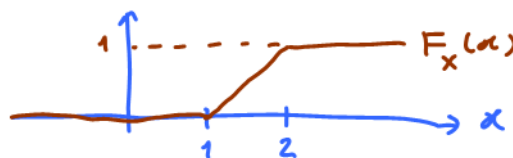
Figure 23: The pdf and cdf for the uniform random variable. [16, Fig. 3.5]

**Example 10.46** (F2011). Suppose  $X$  is uniformly distributed on the interval  $(1, 2)$ . ( $X \sim \mathcal{U}(1, 2)$ .)

(a) Plot the pdf  $f_X(x)$  of  $X$ .



(b) Plot the cdf  $F_X(x)$  of  $X$ .

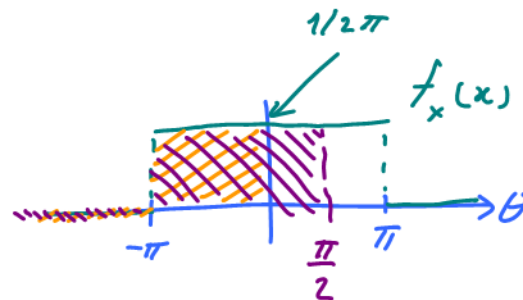


**10.47.** The uniform distribution provides a probability model for selecting a point at random from the interval  $[a, b]$ .

- Use with caution to model a quantity that is known to vary randomly between  $a$  and  $b$  but about which little else is known.

**Example 10.48.** [9, Ex. 4.1 p. 140-141] In coherent radio communications, the phase difference between the transmitter and the receiver, denoted by  $\Theta$ , is modeled as having a uniform density on  $[-\pi, \pi]$ .

(a)  $P[\Theta \leq 0] = \frac{1}{2}$



(b)  $P[\Theta \leq \frac{\pi}{2}] = \frac{3}{4}$

**Exercise 10.49.** Show that when  $X \sim \mathcal{U}([a, b])$ ,  $\mathbb{E}X = \frac{a+b}{2}$ ,  $\text{Var } X = \frac{(b-a)^2}{12}$ , and  $\mathbb{E}[X^2] = \frac{1}{3}(b^2 + ab + a^2)$ .

### 10.4.2 Gaussian Distribution

**10.50.** This is the most widely used model for the distribution of a random variable. When you have many independent random variables, a fundamental result called the central limit theorem (CLT) (informally) says that the sum (or the average) of them can often be approximated by normal distribution.

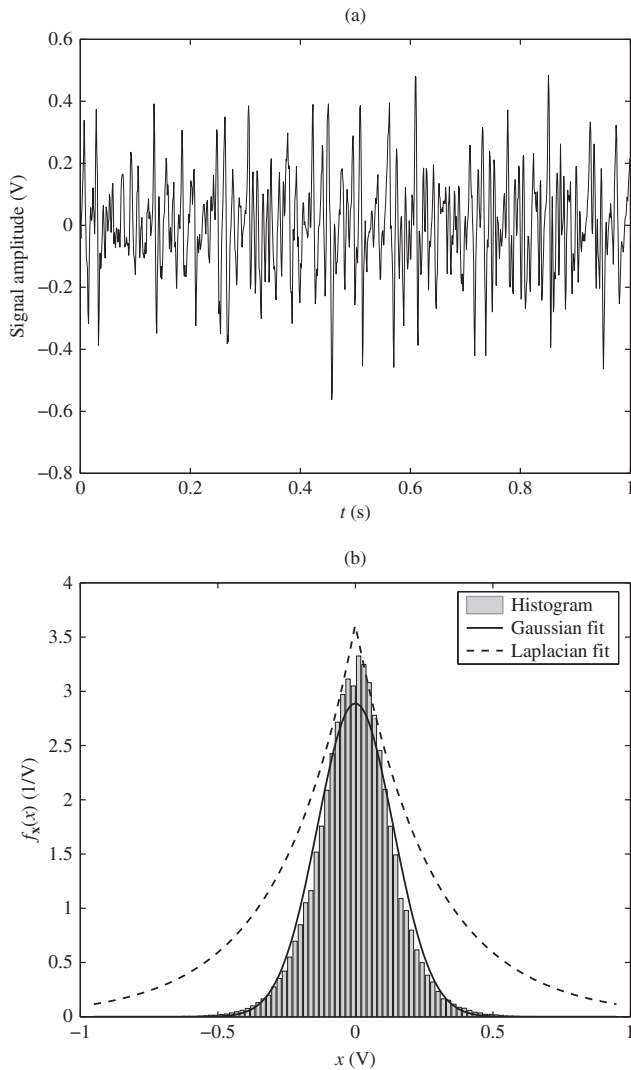


Figure 24: Electrical activity of a skeletal muscle: (a) A sample skeletal muscle (emg) signal, and (b) its histogram and pdf fits. [16, Fig. 3.14]

**Definition 10.51.** *Gaussian* random variables:

- Often called *normal* random variables because they occur so frequently in practice.

- The Gaussian distribution is denoted by  $\mathcal{N}(m, \sigma^2)$ . It has two parameters:  $m \in \mathbb{R}$  and  $\sigma > 0$ .
  - Caution: The second argument in  $\mathcal{N}(m, \sigma^2)$  is  $\sigma^2$  (not  $\sigma$ ).
  - Several references use  $\mu$  instead of  $m$ .
- This family is characterized by pdf of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}.$$

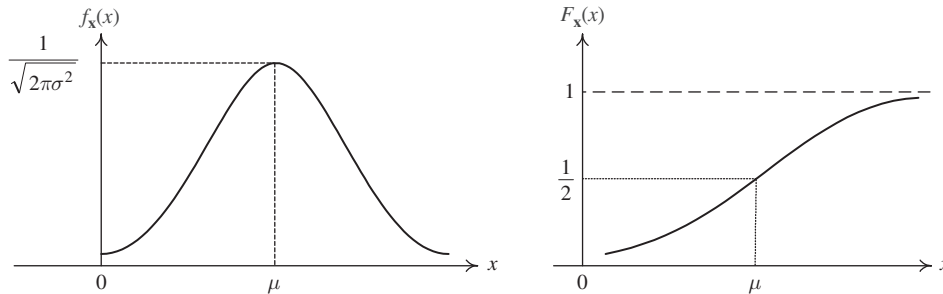


Figure 25: The pdf and cdf of  $\mathcal{N}(\mu, \sigma^2)$ . [16, Fig. 3.6]

- In Excel, use `NORMDIST(x,m,σ, FALSE)`.  
In MATLAB, use `normpdf(x,m,σ)` or `pdf('Normal',x,m,σ)`.
- Figure 25 and Figure 27 display the famous ***bell-shaped*** graphs of the Gaussian pdf. These curves are also called the *normal* curves.
- In MATLAB, use `X = random('Normal',m,σ)` or `X = σ*randn + m` to generate  $X \sim \mathcal{N}(m, \sigma^2)$ .
- $F_X(x)$  has no closed-form expression. However, see 10.58.
  - In MATLAB, use `normcdf(x,m,σ)` or `cdf('Normal',x,m,σ)`.
  - In Excel, use `NORMDIST(x,m,σ, TRUE)`.

**10.52.**  $\mathbb{E}X = m$  and  $\text{Var } X = \sigma^2$ .

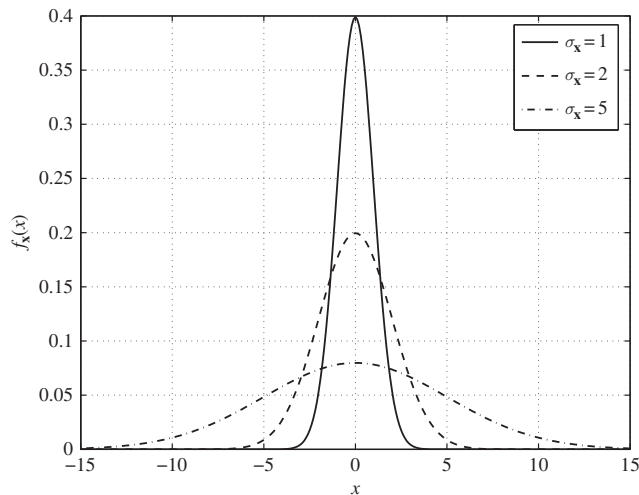


Figure 26: Plots of the zero-mean Gaussian pdf for different values of standard deviation,  $\sigma_X$ . [16, Fig. 3.15]

**10.53.** Important probabilities:

$$P[|X - \mu| < \sigma] = 0.6827;$$

$$P[|X - \mu| > \sigma] = 0.3173;$$

$$P[|X - \mu| > 2\sigma] = 0.0455;$$

$$P[|X - \mu| < 2\sigma] = 0.9545$$

These values are illustrated in Figure 27. We will see these numbers again in Example 10.59.

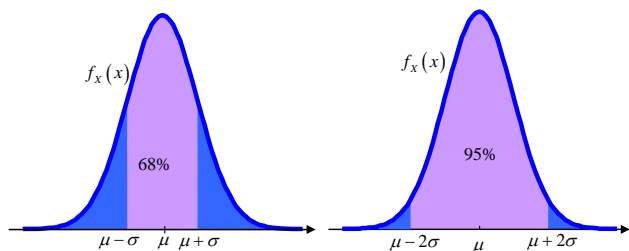


Figure 27: Probability density function of  $X \sim \mathcal{N}(\mu, \sigma^2)$ . The purple areas correspond to  $P[|X - \mu| < \sigma] = 0.6827$  and  $P[|X - \mu| < 2\sigma] = 0.9545$ , respectively.

**Example 10.54.** Figure 28 compares several deviation scores and the normal distribution.

- (a) Standard scores have a mean of zero and a standard deviation of 1.0.
- (b) Scholastic Aptitude Test scores have a mean of 500 and a standard deviation of 100.

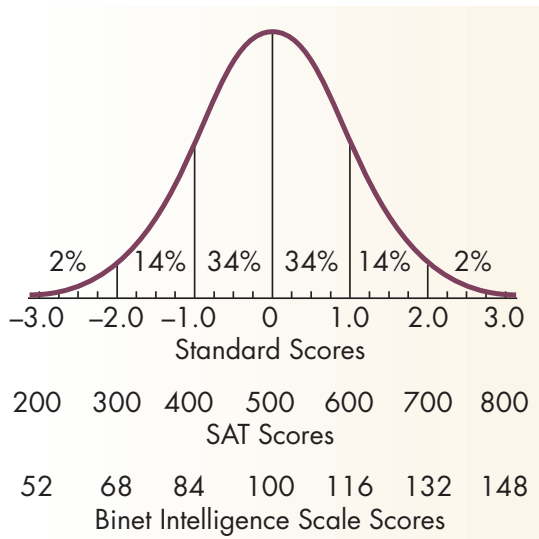


Figure 28: Comparison of Several Deviation Scores and the Normal Distribution

(c) Binet Intelligence Scale<sup>45</sup> scores have a mean of 100 and a standard deviation of 16.

In each case there are 34 percent of the scores between the mean and one standard deviation, 14 percent between one and two standard deviations, and 2 percent beyond two standard deviations. [Source: Beck, *Applying Psychology: Critical and Creative Thinking*.]

**10.55.** The area under a normal probability density function beyond  $3\sigma$  from the mean is quite small. In fact,

$$P[|X - \mu| < 3\sigma] \approx 0.9973.$$

Therefore, approximately 99.73% of the probability of a normal distribution is within the interval  $(\mu - 3\sigma, \mu + 3\sigma)$ .

<sup>45</sup>Alfred Binet, who devised the first general aptitude test at the beginning of the 20th century, defined intelligence as the ability to make adaptations. The general purpose of the test was to determine which children in Paris could benefit from school. Binet's test, like its subsequent revisions, consists of a series of progressively more difficult tasks that children of different ages can successfully complete. A child who can solve problems typically solved by children at a particular age level is said to have that mental age. For example, if a child can successfully do the same tasks that an average 8-year-old can do, he or she is said to have a mental age of 8. The intelligence quotient, or IQ, is defined by the formula:

$$IQ = 100 \times (\text{Mental Age} / \text{Chronological Age})$$

There has been a great deal of controversy in recent years over what intelligence tests measure. Many of the test items depend on either language or other specific cultural experiences for correct answers. Nevertheless, such tests can rather effectively predict school success. If school requires language and the tests measure language ability at a particular point of time in a child's life, then the test is a better-than-chance predictor of school performance.

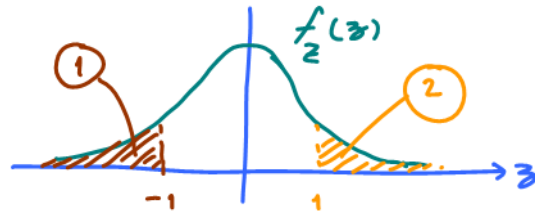
**Definition 10.56.**  $\mathcal{N}(0, 1)$  is the **standard** Gaussian (normal) distribution.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$F_Z(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

- We usually use  $Z$  to denote standard Gaussian RV.
- In Excel, use `NORMSINV(RAND())`.  
In MATLAB, use `randn`.
- The standard normal cdf is denoted by  $\Phi(z)$ .  $\Phi(z) = F_Z(z)$  when  $z \sim \mathcal{N}(0, 1)$ 
  - It inherits all properties of cdf.
  - Moreover, note that  $\Phi(-z) = 1 - \Phi(z)$ .



$$\Phi(-1) = P[Z \leq -1] = \textcircled{1} = \textcircled{2} = 1 - F_Z(1) = 1 - \Phi(1)$$

**10.57.** Relationship between  $\mathcal{N}(0, 1)$  and  $\mathcal{N}(m, \sigma^2)$ .

(a) An arbitrary Gaussian random variable with mean  $m$  and variance  $\sigma^2$  can be represented as  $\sigma Z + m$ , where  $Z \sim \mathcal{N}(0, 1)$ .

This relationship can be used to generate general Gaussian RV from standard Gaussian RV.

(b) If  $X \sim \mathcal{N}(m, \sigma^2)$ , the random variable

$$Z = \frac{X - m}{\sigma}$$

is a standard normal random variable. That is,  $Z \sim \mathcal{N}(0, 1)$ .

- Creating a new random variable by this transformation is referred to as **standardizing**.
- The standardized variable is called “**standard score**” or “**z-score**”.

**10.58.** It is impossible to express the integral of a Gaussian PDF between non-infinite limits (e.g., (20)) as a function that appears on most scientific calculators.



Review:

Finding probability involving  $X \sim \mathcal{N}(m, \sigma^2)$

① There is no closed-form simplification for  $\int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$ .  
(except for some special cases)

② We have a table which gives  $\Phi(z) = F_z(z)$  when  $z \sim \mathcal{N}(0,1)$ ,  
for  $z \in [0, 3)$ .  
The cdf of a standard Gaussian RV

We have shown that  $\Phi(-z) = 1 - \Phi(z)$ . Therefore, the table also gives  $\Phi(z)$  for  $z \in (-3, 0)$ .

For  $|z| > 3, \dots$

③ Therefore, for  $Z \sim \mathcal{N}(0,1)$ ,

$$P[a < Z < b] = \Phi(b) - \Phi(a).$$

④ For general  $X \sim \mathcal{N}(m, \sigma^2)$ ,  $F_x(x) = \Phi\left(\frac{x-m}{\sigma}\right)$

$$P[a < X < b] = F_x(b) - F_x(a) = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)$$

$$P[X < b] = P[X \leq b] = F_x(b) = \Phi\left(\frac{b-m}{\sigma}\right)$$

$$P[X > a] = 1 - P[X \leq a] = 1 - F_x(a) = 1 - \Phi\left(\frac{a-m}{\sigma}\right)$$

- An old but still popular technique to find integrals of the Gaussian PDF is to refer to tables that have been obtained by numerical integration.
  - An example of such table is Table 4, which lists  $\Phi(z)$  for many values of positive  $z$ .
  - For  $X \sim \mathcal{N}(m, \sigma^2)$ , we can show that the CDF of  $X$  can be calculated from

$$F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right)$$

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2} dt = \int_{-\infty}^{\frac{x-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2} d\mu = \Phi\left(\frac{x-m}{\sigma}\right)$$

Let  $\mu = \frac{t-m}{\sigma} \Rightarrow d\mu = \frac{1}{\sigma} dt$

**Example 10.59.** Suppose  $Z \sim \mathcal{N}(0, 1)$ . Evaluate the following probabilities.  $P[a \leq X \leq b] = F_X(b) - F_X(a)$

(a)  $P[-1 \leq Z \leq 1] = F_Z(1) - F_Z(-1) = \Phi(1) - \Phi(-1)$   
 $= 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6826$

(b)  $P[-2 \leq Z \leq 2] = 2\Phi(2) - 1 = 0.9545$   
 $0.97725$

**Example 10.60.** Suppose  $X \sim \mathcal{N}(1, 2)$ . Find  $P[1 \leq X \leq 2]$ .

$$= F_X(2) - F_X(1) = \Phi\left(\frac{2-1}{\sqrt{2}}\right) - \Phi\left(\frac{1-1}{\sqrt{2}}\right) = \Phi\left(\frac{1}{\sqrt{2}}\right) - \Phi(0)$$

$x \sim \mathcal{N}(1, 2)$        $\sigma^2 \Rightarrow \sigma = \sqrt{2}$

from the  $\Phi$  table       $0.7071$        $0.5$

$\approx 0.7071 - 0.5 = 0.2071$

**Example 10.61.** Signal Detection: Assume that in the detection of a digital signal, the background noise follows a normal distribution with a mean of 0 volt and standard deviation of 0.45 volt. The system assumes a digital 1 has been transmitted when the

$$\Phi(-0.5) = 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085$$

| z    | $\Phi(z)$ | z    | $\Phi(z)$ | z    | $\Phi(z)$ | z    | $\Phi(z)$ | z    | $\Phi(z)$ | z    | $\Phi(z)$ |
|------|-----------|------|-----------|------|-----------|------|-----------|------|-----------|------|-----------|
| 0.00 | 0.5000    | 0.50 | 0.6915    | 1.00 | 0.8413    | 1.50 | 0.9332    | 2.00 | 0.97725   | 2.50 | 0.99379   |
| 0.01 | 0.5040    | 0.51 | 0.6950    | 1.01 | 0.8438    | 1.51 | 0.9345    | 2.01 | 0.97778   | 2.51 | 0.99396   |
| 0.02 | 0.5080    | 0.52 | 0.6985    | 1.02 | 0.8461    | 1.52 | 0.9357    | 2.02 | 0.97831   | 2.52 | 0.99413   |
| 0.03 | 0.5120    | 0.53 | 0.7019    | 1.03 | 0.8485    | 1.53 | 0.9370    | 2.03 | 0.97882   | 2.53 | 0.99430   |
| 0.04 | 0.5160    | 0.54 | 0.7054    | 1.04 | 0.8508    | 1.54 | 0.9382    | 2.04 | 0.97932   | 2.54 | 0.99446   |
| 0.05 | 0.5199    | 0.55 | 0.7088    | 1.05 | 0.8531    | 1.55 | 0.9394    | 2.05 | 0.97982   | 2.55 | 0.99461   |
| 0.06 | 0.5239    | 0.56 | 0.7123    | 1.06 | 0.8554    | 1.56 | 0.9406    | 2.06 | 0.98030   | 2.56 | 0.99477   |
| 0.07 | 0.5279    | 0.57 | 0.7157    | 1.07 | 0.8577    | 1.57 | 0.9418    | 2.07 | 0.98077   | 2.57 | 0.99492   |
| 0.08 | 0.5319    | 0.58 | 0.7190    | 1.08 | 0.8599    | 1.58 | 0.9429    | 2.08 | 0.98124   | 2.58 | 0.99506   |
| 0.09 | 0.5359    | 0.59 | 0.7224    | 1.09 | 0.8621    | 1.59 | 0.9441    | 2.09 | 0.98169   | 2.59 | 0.99520   |
| 0.10 | 0.5398    | 0.60 | 0.7257    | 1.10 | 0.8643    | 1.60 | 0.9452    | 2.10 | 0.98214   | 2.60 | 0.99534   |
| 0.11 | 0.5438    | 0.61 | 0.7291    | 1.11 | 0.8665    | 1.61 | 0.9463    | 2.11 | 0.98257   | 2.61 | 0.99547   |
| 0.12 | 0.5478    | 0.62 | 0.7324    | 1.12 | 0.8686    | 1.62 | 0.9474    | 2.12 | 0.98300   | 2.62 | 0.99560   |
| 0.13 | 0.5517    | 0.63 | 0.7357    | 1.13 | 0.8708    | 1.63 | 0.9484    | 2.13 | 0.98341   | 2.63 | 0.99573   |
| 0.14 | 0.5557    | 0.64 | 0.7389    | 1.14 | 0.8729    | 1.64 | 0.9495    | 2.14 | 0.98382   | 2.64 | 0.99585   |
| 0.15 | 0.5596    | 0.65 | 0.7422    | 1.15 | 0.8749    | 1.65 | 0.9505    | 2.15 | 0.98422   | 2.65 | 0.99598   |
| 0.16 | 0.5636    | 0.66 | 0.7454    | 1.16 | 0.8770    | 1.66 | 0.9515    | 2.16 | 0.98461   | 2.66 | 0.99609   |
| 0.17 | 0.5675    | 0.67 | 0.7486    | 1.17 | 0.8790    | 1.67 | 0.9525    | 2.17 | 0.98500   | 2.67 | 0.99621   |
| 0.18 | 0.5714    | 0.68 | 0.7517    | 1.18 | 0.8810    | 1.68 | 0.9535    | 2.18 | 0.98537   | 2.68 | 0.99632   |
| 0.19 | 0.5753    | 0.69 | 0.7549    | 1.19 | 0.8830    | 1.69 | 0.9545    | 2.19 | 0.98574   | 2.69 | 0.99643   |
| 0.20 | 0.5793    | 0.70 | 0.7580    | 1.20 | 0.8849    | 1.70 | 0.9554    | 2.20 | 0.98610   | 2.70 | 0.99653   |
| 0.21 | 0.5832    | 0.71 | 0.7611    | 1.21 | 0.8869    | 1.71 | 0.9564    | 2.21 | 0.98645   | 2.71 | 0.99664   |
| 0.22 | 0.5871    | 0.72 | 0.7642    | 1.22 | 0.8888    | 1.72 | 0.9573    | 2.22 | 0.98679   | 2.72 | 0.99674   |
| 0.23 | 0.5910    | 0.73 | 0.7673    | 1.23 | 0.8907    | 1.73 | 0.9582    | 2.23 | 0.98713   | 2.73 | 0.99683   |
| 0.24 | 0.5948    | 0.74 | 0.7704    | 1.24 | 0.8925    | 1.74 | 0.9591    | 2.24 | 0.98745   | 2.74 | 0.99693   |
| 0.25 | 0.5987    | 0.75 | 0.7734    | 1.25 | 0.8944    | 1.75 | 0.9599    | 2.25 | 0.98778   | 2.75 | 0.99702   |
| 0.26 | 0.6026    | 0.76 | 0.7764    | 1.26 | 0.8962    | 1.76 | 0.9608    | 2.26 | 0.98809   | 2.76 | 0.99711   |
| 0.27 | 0.6064    | 0.77 | 0.7794    | 1.27 | 0.8980    | 1.77 | 0.9616    | 2.27 | 0.98840   | 2.77 | 0.99720   |
| 0.28 | 0.6103    | 0.78 | 0.7823    | 1.28 | 0.8997    | 1.78 | 0.9625    | 2.28 | 0.98870   | 2.78 | 0.99728   |
| 0.29 | 0.6141    | 0.79 | 0.7852    | 1.29 | 0.9015    | 1.79 | 0.9633    | 2.29 | 0.98899   | 2.79 | 0.99736   |
| 0.30 | 0.6179    | 0.80 | 0.7881    | 1.30 | 0.9032    | 1.80 | 0.9641    | 2.30 | 0.98928   | 2.80 | 0.99744   |
| 0.31 | 0.6217    | 0.81 | 0.7910    | 1.31 | 0.9049    | 1.81 | 0.9649    | 2.31 | 0.98956   | 2.81 | 0.99752   |
| 0.32 | 0.6255    | 0.82 | 0.7939    | 1.32 | 0.9066    | 1.82 | 0.9656    | 2.32 | 0.98983   | 2.82 | 0.99760   |
| 0.33 | 0.6293    | 0.83 | 0.7967    | 1.33 | 0.9082    | 1.83 | 0.9664    | 2.33 | 0.99010   | 2.83 | 0.99767   |
| 0.34 | 0.6331    | 0.84 | 0.7995    | 1.34 | 0.9099    | 1.84 | 0.9671    | 2.34 | 0.99036   | 2.84 | 0.99774   |
| 0.35 | 0.6368    | 0.85 | 0.8023    | 1.35 | 0.9115    | 1.85 | 0.9678    | 2.35 | 0.99061   | 2.85 | 0.99781   |
| 0.36 | 0.6406    | 0.86 | 0.8051    | 1.36 | 0.9131    | 1.86 | 0.9686    | 2.36 | 0.99086   | 2.86 | 0.99788   |
| 0.37 | 0.6443    | 0.87 | 0.8078    | 1.37 | 0.9147    | 1.87 | 0.9693    | 2.37 | 0.99111   | 2.87 | 0.99795   |
| 0.38 | 0.6480    | 0.88 | 0.8106    | 1.38 | 0.9162    | 1.88 | 0.9699    | 2.38 | 0.99134   | 2.88 | 0.99801   |
| 0.39 | 0.6517    | 0.89 | 0.8133    | 1.39 | 0.9177    | 1.89 | 0.9706    | 2.39 | 0.99158   | 2.89 | 0.99807   |
| 0.40 | 0.6554    | 0.90 | 0.8159    | 1.40 | 0.9192    | 1.90 | 0.9713    | 2.40 | 0.99180   | 2.90 | 0.99813   |
| 0.41 | 0.6591    | 0.91 | 0.8186    | 1.41 | 0.9207    | 1.91 | 0.9719    | 2.41 | 0.99202   | 2.91 | 0.99819   |
| 0.42 | 0.6628    | 0.92 | 0.8212    | 1.42 | 0.9222    | 1.92 | 0.9726    | 2.42 | 0.99224   | 2.92 | 0.99825   |
| 0.43 | 0.6664    | 0.93 | 0.8238    | 1.43 | 0.9236    | 1.93 | 0.9732    | 2.43 | 0.99245   | 2.93 | 0.99831   |
| 0.44 | 0.6700    | 0.94 | 0.8264    | 1.44 | 0.9251    | 1.94 | 0.9738    | 2.44 | 0.99266   | 2.94 | 0.99836   |
| 0.45 | 0.6736    | 0.95 | 0.8289    | 1.45 | 0.9265    | 1.95 | 0.9744    | 2.45 | 0.99286   | 2.95 | 0.99841   |
| 0.46 | 0.6772    | 0.96 | 0.8315    | 1.46 | 0.9279    | 1.96 | 0.9750    | 2.46 | 0.99305   | 2.96 | 0.99846   |
| 0.47 | 0.6808    | 0.97 | 0.8340    | 1.47 | 0.9292    | 1.97 | 0.9756    | 2.47 | 0.99324   | 2.97 | 0.99851   |
| 0.48 | 0.6844    | 0.98 | 0.8365    | 1.48 | 0.9306    | 1.98 | 0.9761    | 2.48 | 0.99343   | 2.98 | 0.99856   |
| 0.49 | 0.6879    | 0.99 | 0.8389    | 1.49 | 0.9319    | 1.99 | 0.9767    | 2.49 | 0.99361   | 2.99 | 0.99861   |

Table 4: The standard normal CDF:  $\Phi(z)$

$$N \sim \mathcal{N}(0, 0.45^2)$$

$$P[\text{detecting a } 1^c] = P[N > 0.9] = 1 - P[N \leq 0.9] = 1 - F_N(0.9)$$

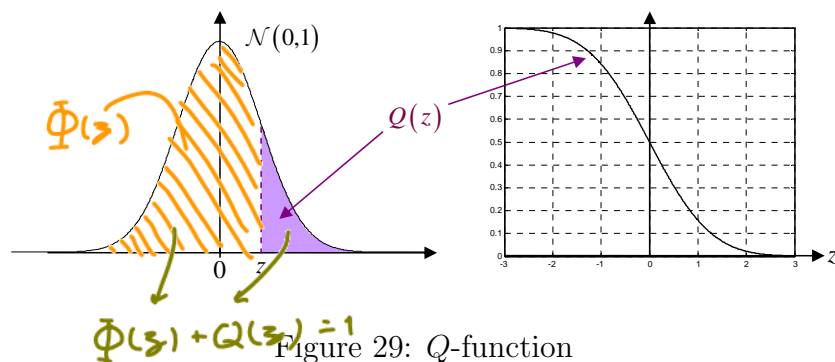
(assuming nothing was sent)      noise

$$= 1 - \Phi\left(\frac{0.9 - 0}{0.45}\right) = 1 - \Phi(2) = 1 - 0.97725 = 0.02275$$

voltage exceeds 0.9. (Otherwise, it assumes a digital 0 has been transmitted)

What is the probability of detecting a digital 1 when none was sent? [Montgomery and Runger, 2013, Ex. 4-15]

**10.62. Q-function:**  $Q(z) = \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$  corresponds to  $P[Z > z]$  where  $Z \sim \mathcal{N}(0, 1)$ ; that is  $Q(z)$  is the probability of the “tail” of  $\mathcal{N}(0, 1)$ . The  $Q$  function is then a **complementary cdf (ccdf)**.



- $Q$  is a decreasing function with  $Q(0) = \frac{1}{2}$ .
- $Q(-z) = 1 - Q(z) = \Phi(z)$
- Table 5 lists the values of  $Q(z)$  for  $z$  between 3 to 5.
  - For  $z$  between 0 to 3, we use  $Q(z) = 1 - \Phi(z)$ .
  - For  $z \geq 5$ , the value of  $Q(z)$  is extremely small. We may assume  $Q(z) \approx 0$ .

**10.63. Error function (MATLAB):**  $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx = 1 - 2Q(\sqrt{2}z)$

- It is an odd function of  $z$ .
- For  $z \geq 0$ , it corresponds to  $P[|X| < z]$  where  $X \sim \mathcal{N}(0, \frac{1}{2})$ .
- $\lim_{z \rightarrow \infty} \text{erf}(z) = 1$

| z    | Q(z)     | z    | Q(z)     | z    | Q(z)     | z    | Q(z)     | z    | Q(z)     |
|------|----------|------|----------|------|----------|------|----------|------|----------|
| 3.00 | 1.35E-03 | 3.40 | 3.37E-04 | 3.80 | 7.23E-05 | 4.20 | 1.33E-05 | 4.60 | 2.11E-06 |
| 3.01 | 1.31E-03 | 3.41 | 3.25E-04 | 3.81 | 6.95E-05 | 4.21 | 1.28E-05 | 4.61 | 2.01E-06 |
| 3.02 | 1.26E-03 | 3.42 | 3.13E-04 | 3.82 | 6.67E-05 | 4.22 | 1.22E-05 | 4.62 | 1.92E-06 |
| 3.03 | 1.22E-03 | 3.43 | 3.02E-04 | 3.83 | 6.41E-05 | 4.23 | 1.17E-05 | 4.63 | 1.83E-06 |
| 3.04 | 1.18E-03 | 3.44 | 2.91E-04 | 3.84 | 6.15E-05 | 4.24 | 1.12E-05 | 4.64 | 1.74E-06 |
| 3.05 | 1.14E-03 | 3.45 | 2.80E-04 | 3.85 | 5.91E-05 | 4.25 | 1.07E-05 | 4.65 | 1.66E-06 |
| 3.06 | 1.11E-03 | 3.46 | 2.70E-04 | 3.86 | 5.67E-05 | 4.26 | 1.02E-05 | 4.66 | 1.58E-06 |
| 3.07 | 1.07E-03 | 3.47 | 2.60E-04 | 3.87 | 5.44E-05 | 4.27 | 9.77E-06 | 4.67 | 1.51E-06 |
| 3.08 | 1.04E-03 | 3.48 | 2.51E-04 | 3.88 | 5.22E-05 | 4.28 | 9.34E-06 | 4.68 | 1.43E-06 |
| 3.09 | 1.00E-03 | 3.49 | 2.42E-04 | 3.89 | 5.01E-05 | 4.29 | 8.93E-06 | 4.69 | 1.37E-06 |
| 3.10 | 9.68E-04 | 3.50 | 2.33E-04 | 3.90 | 4.81E-05 | 4.30 | 8.54E-06 | 4.70 | 1.30E-06 |
| 3.11 | 9.35E-04 | 3.51 | 2.24E-04 | 3.91 | 4.61E-05 | 4.31 | 8.16E-06 | 4.71 | 1.24E-06 |
| 3.12 | 9.04E-04 | 3.52 | 2.16E-04 | 3.92 | 4.43E-05 | 4.32 | 7.80E-06 | 4.72 | 1.18E-06 |
| 3.13 | 8.74E-04 | 3.53 | 2.08E-04 | 3.93 | 4.25E-05 | 4.33 | 7.46E-06 | 4.73 | 1.12E-06 |
| 3.14 | 8.45E-04 | 3.54 | 2.00E-04 | 3.94 | 4.07E-05 | 4.34 | 7.12E-06 | 4.74 | 1.07E-06 |
| 3.15 | 8.16E-04 | 3.55 | 1.93E-04 | 3.95 | 3.91E-05 | 4.35 | 6.81E-06 | 4.75 | 1.02E-06 |
| 3.16 | 7.89E-04 | 3.56 | 1.85E-04 | 3.96 | 3.75E-05 | 4.36 | 6.50E-06 | 4.76 | 9.68E-07 |
| 3.17 | 7.62E-04 | 3.57 | 1.78E-04 | 3.97 | 3.59E-05 | 4.37 | 6.21E-06 | 4.77 | 9.21E-07 |
| 3.18 | 7.36E-04 | 3.58 | 1.72E-04 | 3.98 | 3.45E-05 | 4.38 | 5.93E-06 | 4.78 | 8.76E-07 |
| 3.19 | 7.11E-04 | 3.59 | 1.65E-04 | 3.99 | 3.30E-05 | 4.39 | 5.67E-06 | 4.79 | 8.34E-07 |
| 3.20 | 6.87E-04 | 3.60 | 1.59E-04 | 4.00 | 3.17E-05 | 4.40 | 5.41E-06 | 4.80 | 7.93E-07 |
| 3.21 | 6.64E-04 | 3.61 | 1.53E-04 | 4.01 | 3.04E-05 | 4.41 | 5.17E-06 | 4.81 | 7.55E-07 |
| 3.22 | 6.41E-04 | 3.62 | 1.47E-04 | 4.02 | 2.91E-05 | 4.42 | 4.94E-06 | 4.82 | 7.18E-07 |
| 3.23 | 6.19E-04 | 3.63 | 1.42E-04 | 4.03 | 2.79E-05 | 4.43 | 4.71E-06 | 4.83 | 6.83E-07 |
| 3.24 | 5.98E-04 | 3.64 | 1.36E-04 | 4.04 | 2.67E-05 | 4.44 | 4.50E-06 | 4.84 | 6.49E-07 |
| 3.25 | 5.77E-04 | 3.65 | 1.31E-04 | 4.05 | 2.56E-05 | 4.45 | 4.29E-06 | 4.85 | 6.17E-07 |
| 3.26 | 5.57E-04 | 3.66 | 1.26E-04 | 4.06 | 2.45E-05 | 4.46 | 4.10E-06 | 4.86 | 5.87E-07 |
| 3.27 | 5.38E-04 | 3.67 | 1.21E-04 | 4.07 | 2.35E-05 | 4.47 | 3.91E-06 | 4.87 | 5.58E-07 |
| 3.28 | 5.19E-04 | 3.68 | 1.17E-04 | 4.08 | 2.25E-05 | 4.48 | 3.73E-06 | 4.88 | 5.30E-07 |
| 3.29 | 5.01E-04 | 3.69 | 1.12E-04 | 4.09 | 2.16E-05 | 4.49 | 3.56E-06 | 4.89 | 5.04E-07 |
| 3.30 | 4.83E-04 | 3.70 | 1.08E-04 | 4.10 | 2.07E-05 | 4.50 | 3.40E-06 | 4.90 | 4.79E-07 |
| 3.31 | 4.66E-04 | 3.71 | 1.04E-04 | 4.11 | 1.98E-05 | 4.51 | 3.24E-06 | 4.91 | 4.55E-07 |
| 3.32 | 4.50E-04 | 3.72 | 9.96E-05 | 4.12 | 1.89E-05 | 4.52 | 3.09E-06 | 4.92 | 4.33E-07 |
| 3.33 | 4.34E-04 | 3.73 | 9.57E-05 | 4.13 | 1.81E-05 | 4.53 | 2.95E-06 | 4.93 | 4.11E-07 |
| 3.34 | 4.19E-04 | 3.74 | 9.20E-05 | 4.14 | 1.74E-05 | 4.54 | 2.81E-06 | 4.94 | 3.91E-07 |
| 3.35 | 4.04E-04 | 3.75 | 8.84E-05 | 4.15 | 1.66E-05 | 4.55 | 2.68E-06 | 4.95 | 3.71E-07 |
| 3.36 | 3.90E-04 | 3.76 | 8.50E-05 | 4.16 | 1.59E-05 | 4.56 | 2.56E-06 | 4.96 | 3.52E-07 |
| 3.37 | 3.76E-04 | 3.77 | 8.16E-05 | 4.17 | 1.52E-05 | 4.57 | 2.44E-06 | 4.97 | 3.35E-07 |
| 3.38 | 3.62E-04 | 3.78 | 7.84E-05 | 4.18 | 1.46E-05 | 4.58 | 2.32E-06 | 4.98 | 3.18E-07 |
| 3.39 | 3.49E-04 | 3.79 | 7.53E-05 | 4.19 | 1.39E-05 | 4.59 | 2.22E-06 | 4.99 | 3.02E-07 |

Table 5: The standard normal complementary CDF:  $Q(z)$

(d)  $\operatorname{erf}(-z) = -\operatorname{erf}(z)$

(e)  $\Phi(x) = \frac{1}{2} \left( 1 + \operatorname{erf} \left( \frac{x}{\sqrt{2}} \right) \right) = \frac{1}{2} \operatorname{erfc} \left( -\frac{x}{\sqrt{2}} \right)$

(f) The complementary error function:

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = 2Q(\sqrt{2}z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-x^2} dx$$

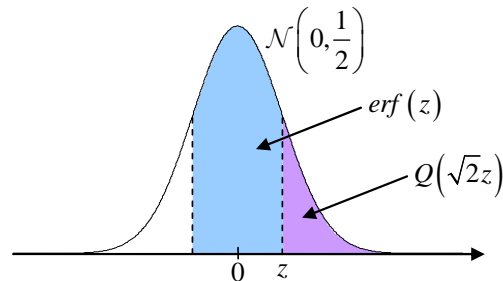


Figure 30: erf-function and Q-function

### 10.4.3 Exponential Distribution

**Definition 10.64.** The exponential distribution is denoted by  $\mathcal{E}(\lambda)$ .

(a)  $\lambda > 0$  is a parameter of the distribution, often called the **rate parameter**.

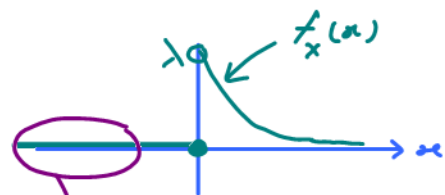
(b) Characterized by

- $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$
- $F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$

$$P[X \leq x] = \int_{-\infty}^x f_X(t) dt$$

When  $x > 0$ ,

$$F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_0^x \lambda e^{-\lambda t} dt = \lambda \times \left( \frac{1}{-\lambda} \right) e^{-\lambda t} \Big|_0^x = 1 - e^{-\lambda x}$$



(c) MATLAB:

- $X = \text{exprnd}(1/\lambda)$  or  $\text{random}('exp', 1/\lambda)$
- $f_X(x) = \text{exp pdf}(x, 1/\lambda)$  or  $\text{pdf}('exp', x, 1/\lambda)$
- $F_X(x) = \text{expcdf}(x, 1/\lambda)$  or  $\text{cdf}('exp', x, 1/\lambda)$

**10.65.** The exponential distribution is intimately related to the **Poisson process**. In fact, the random variable  $X$  that equals **the “distance”** (or length or duration) **between (any) successive events of a Poisson process** with parameter  $\lambda$  is an exponential random variable with the same parameter.

**Example 10.66.** Exponential distribution is often used as a probability model for the (waiting) time until the next “rare” event occurs.

- time elapsed until the next earthquake in a certain region
- decay time of a radioactive particle
- time between independent events such as arrivals at a service facility or arrivals of customers in a shop.
- duration of a cell-phone call
- time it takes a computer network to transmit a message from one node to another.

**10.67.** In Example 10.35, we showed that  $\mathbb{E}X = \frac{1}{\lambda}$ .

**Example 10.68.** Suppose  $X \sim \mathcal{E}(\lambda)$ , find  $P[1 < X < 2]$ .

Method ①:  $P[1 < X < 2] = F_X(2) - F_X(1) = (1 - e^{-\lambda(2)}) - (1 - e^{-\lambda(1)}) = e^{-\lambda} - e^{-2\lambda}$

Method ②:  $P[1 < X < 2] = \int_1^2 f_X(x) dx = \int_1^2 \lambda e^{-\lambda x} dx = \left. \frac{\lambda e^{-\lambda x}}{-\lambda} \right|_1^2 = e^{-\lambda} - e^{-2\lambda}$

**10.69.** Survival-, survivor-, or reliability-function:

$\uparrow$   
complementary  
 $\text{ccdf} \equiv 1 - \text{cdf} = 1 - F_X(x) = 1 - P[X \leq x] = P[X > x]$

**Exercise 10.70.** Exponential random variable as a continuous version of geometric random variable: Suppose  $X \sim \mathcal{E}(\lambda)$ . Show that  $\lfloor X \rfloor \sim \mathcal{G}_0(e^{-\lambda})$  and  $\lceil X \rceil \sim \mathcal{G}_1(e^{-\lambda})$

**Example 10.71.** Phone Company A charges \$0.15 per minute for telephone calls. For any fraction of a minute at the end of a call, they charge for a full minute. Phone Company B also charges \$0.15 per minute. However, Phone Company B calculates its charge based on the exact duration of a call. If  $T$ , the duration of a call in minutes, is exponential with parameter  $\lambda = 1/3$ , what are the expected revenues per call  $\mathbb{E}[R_A]$  and  $\mathbb{E}[R_B]$  for companies A and B?

**Solution:** First, note that  $\mathbb{E}T = \frac{1}{\lambda} = 3$ . Hence,

$$\mathbb{E}[R_B] = \mathbb{E}[0.15 \times T] = 0.15\mathbb{E}T = \$0.45.$$

and

$$\mathbb{E}[R_A] = \mathbb{E}[0.15 \times \lceil T \rceil] = 0.15\mathbb{E}[\lceil T \rceil].$$

Now, recall, from Exercise 10.70, that  $\lceil T \rceil \sim \mathcal{G}_1(e^{-\lambda})$ . Hence,  $\mathbb{E}[\lceil T \rceil] = \frac{1}{1-e^{-\lambda}} \approx 3.53$ . Therefore,

$$\mathbb{E}[R_A] = 0.15\mathbb{E}[\lceil T \rceil] \approx 0.5292.$$

**10.72. Memoryless property:** The exponential r.v. is the only continuous<sup>46</sup> r.v. on  $[0, \infty)$  that satisfies the memoryless property:

$$P[X > s + x | X > s] = P[X > x]$$

for all  $x > 0$  and all  $s > 0$  [18, p. 157–159]. In words, the future is independent of the past. The fact that it hasn't happened yet, tells us nothing about how much longer it will take before it does happen.

- Imagining that the exponentially distributed random variable  $X$  represents the lifetime of an item, the residual life of an item has the same exponential distribution as the original lifetime, regardless of how long the item has been already in use. In other words, there is no deterioration/degradation over time. If it is still currently working after 20 years of use, then today, its condition is “just like new”.

---

<sup>46</sup>For discrete random variable, geometric random variables satisfy the memoryless property.



- In particular, suppose we define the set  $B+x$  to be  $\{x + b : b \in B\}$ . For any  $x > 0$  and set  $B \subset [0, \infty)$ , we have

$$P[X \in B + x | X > x] = P[X \in B]$$

because

$$\frac{P[X \in B + x]}{P[X > x]} = \frac{\int_{B+x} \lambda e^{-\lambda t} dt}{e^{-\lambda x}} \stackrel{\tau=t-x}{=} \frac{\int_B \lambda e^{-\lambda(\tau+x)} d\tau}{e^{-\lambda x}}.$$

**Example 10.73.** The exponential distribution is often used in reliability studies as the model for the time until failure of a device. For example, the lifetime of a semiconductor chip might be modeled as an exponential random variable with a mean of 40,000 hours.

The lack of memory property of the exponential distribution implies that the device does not wear out. That is, regardless of how long the device has been operating, the probability of a failure in the next 1000 hours is the same as the probability of a failure in the first 1000 hours of operation.

**10.74.** The lifetime  $L$  of a device with failures caused by random shocks might be appropriately modeled as an exponential random variable. However, the lifetime  $L$  of a device that suffers slow mechanical wear, such as bearing wear, is better modeled by other distributions such as the Weibull distribution.

**10.75.** Summary:

| $X \sim$                                     | Support $S_X$ | $f_X(x) =$                                                                        | $\mathbb{E}X$ | $\text{Var} X$ |
|----------------------------------------------|---------------|-----------------------------------------------------------------------------------|---------------|----------------|
| Uniform $\mathcal{U}(a, b)$                  | $(a, b)$      | $\begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$ |               |                |
| Normal (Gaussian) $\mathcal{N}(m, \sigma^2)$ | $\mathbb{R}$  | $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$   | $m$           | $\sigma^2$     |
| Exponential $\mathcal{E}(\lambda)$           | $(0, \infty)$ | $\begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$     |               |                |

Table 6: Examples of probability density functions. Here,  $\lambda, \sigma > 0$ .

continuous RV

### 10.5 Function of Continuous Random Variables: SISO

Reconsider the derived random variable  $Y = g(X)$ .

$$x \rightarrow \boxed{g(\cdot)} \rightarrow Y = g(x)$$

$$Y = X^2$$

$$\mathbb{E}[Y] = \mathbb{E}[X^2]$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

Recall that we can find  $\mathbb{E}Y$  easily by (22):

$$\mathbb{E}Y = \mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f_X(x) dx.$$

However, there are cases when we have to evaluate probability directly involving the random variable  $Y$  or find  $f_Y(y)$  directly.

Recall that for discrete random variables, it is easy to find  $p_Y(y)$  by adding all  $p_X(x)$  over all  $x$  such that  $g(x) = y$ :

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x). \quad (23)$$

For continuous random variables, it turns out that we can't<sup>47</sup> simply integrate or add the pdf of  $X$  to get the pdf of  $Y$ .

**10.76.** For  $Y = g(X)$ , if you want to find  $f_Y(y)$ , the following *two-step procedure* will always work and is easy to remember:

- (a) Find the cdf  $F_Y(y) = P[Y \leq y] = P[g(X) \leq y]$ .  
Check that  $Y$  is a continuous RV.
- (b) Compute the pdf from the cdf by "finding the derivative"  
 $f_Y(y) = \frac{d}{dy} F_Y(y)$  (as described in 10.25).

**Example 10.77.** Suppose  $X \sim \mathcal{E}(\lambda)$ . Let  $Y = 5X$ . Find  $f_Y(y)$ .

step (a):  $F_Y(y) \equiv P[Y \leq y] = P[5X \leq y] = P[X \leq \frac{y}{5}] = F_X(\frac{y}{5})$   
 $X \sim \mathcal{E}(\lambda) \Rightarrow X$  is a continuous RV  $\Rightarrow F_X(x)$  is continuous func. of  $x$   
 $\Rightarrow F_X(\frac{y}{5})$  is continuous function of  $y$ .  $\Rightarrow Y$  is a continuous RV

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

step (b):  $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\frac{y}{5}) = \frac{1}{5} f_X(\frac{y}{5})$   
 chain rule

Chain rule  
 $\frac{d}{dt} g(h(t)) = g'(h(t)) h'(t)$

<sup>47</sup>When you applied Equation (23) to continuous random variables, what you would get is  $0 = 0$ , which is true but not interesting nor useful.

$$Y \sim \mathcal{E}(\frac{\lambda}{5})$$

$$= \begin{cases} \frac{1}{5} \lambda e^{-\lambda \frac{y}{5}}, & \frac{y}{5} > 0 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{\lambda}{5} e^{-\frac{\lambda}{5} y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

**10.78. Linear Transformation:** Suppose  $Y = aX + b$ . Then, the cdf of  $Y$  is given by

$$F_Y(y) = P[Y \leq y] = P[aX + b \leq y] = \begin{cases} P\left[X \leq \frac{y-b}{a}\right], & a > 0, \\ P\left[X \geq \frac{y-b}{a}\right], & a < 0. \end{cases}$$

Now, by definition, we know that

$$P\left[X \leq \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right),$$

and

$$\begin{aligned} P\left[X \geq \frac{y-b}{a}\right] &= P\left[X > \frac{y-b}{a}\right] + P\left[X = \frac{y-b}{a}\right] \\ &= 1 - F_X\left(\frac{y-b}{a}\right) + P\left[X = \frac{y-b}{a}\right]. \end{aligned}$$

For continuous random variable,  $P\left[X = \frac{y-b}{a}\right] = 0$ . Hence,

$$F_Y(y) = \begin{cases} F_X\left(\frac{y-b}{a}\right), & a > 0, \\ 1 - F_X\left(\frac{y-b}{a}\right), & a < 0. \end{cases}$$

Finally, fundamental theorem of calculus and chain rule gives

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} \frac{1}{a}f_X\left(\frac{y-b}{a}\right), & a > 0, \\ -\frac{1}{a}f_X\left(\frac{y-b}{a}\right), & a < 0. \end{cases}$$

Note that we can further simplify the final formula by using the  $|\cdot|$  function:

$$f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right), \quad a \neq 0.$$

In the example,  
 $a=5, b=0,$

$$(24) \quad f_Y(y) = \frac{1}{5}f_X\left(\frac{y-0}{5}\right)$$

Graphically, to get the plots of  $f_Y$ , we compress  $f_X$  horizontally by a factor of  $a$ , scale it vertically by a factor of  $1/|a|$ , and shift it to the right by  $b$ .

Of course, if  $a = 0$ , then we get the uninteresting degenerated random variable  $Y \equiv b$ .

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

10.79. Suppose  $X \sim \mathcal{N}(m, \sigma^2)$  and  $Y = aX + b$  for some constants  $a$  and  $b$ . Then, we can use (24) to show that  $Y \sim \mathcal{N}(am + b, a^2\sigma^2)$ .

$$f_Y(y) = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left(\frac{y-b}{a}-m\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}(|a|\sigma)} e^{-\frac{1}{2}\left(\frac{y-(am+b)}{\sigma|a|}\right)^2}$$

$$\Rightarrow Y \sim \mathcal{N}(am+b, a^2\sigma^2)$$

Note: From Section 10.4,

$$X \sim \mathcal{N}(m, \sigma^2)$$

$$\mathbb{E}X = m, \text{Var} X = \sigma^2$$

Now, we have  $Y = aX + b$ ,

$$\mathbb{E}Y = a\mathbb{E}X + b = am + b$$

$$\text{Var} Y = a^2 \text{Var} X = a^2\sigma^2$$

The new info. here is that we have now shown that  $Y$  is still a Gaussian RV.

Special Cases

①  $Z \sim \mathcal{N}(0,1) \Rightarrow X = \sigma Z + m \sim \mathcal{N}(m, \sigma^2)$

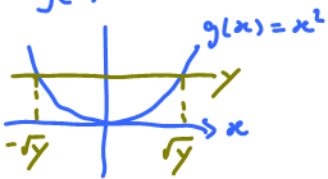
②  $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow Z = \frac{X-m}{\sigma} \sim \mathcal{N}(0,1)$

**Example 10.80.** Amplitude modulation in certain communication systems can be accomplished using various nonlinear devices such as a semiconductor diode. Suppose we model the nonlinear device by the function  $Y = X^2$ . If the input  $X$  is a continuous random variable, find the density of the output  $Y = X^2$ .

①  $F_Y(y) \equiv P[Y \leq y] = P[X^2 \leq y]$

$$Y = g(X)$$

$$g(x) = x^2$$



①.1 Note that  $Y = X^2 \geq 0$ .

Therefore, for  $y < 0$ ,  $F_Y(y) = 0$ .

①.2 For any  $y > 0$ ,

$$F_Y(y) = P[-\sqrt{y} < X < \sqrt{y}] = F_X(b) - F_X(a)$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

①.3 For  $y = 0$ ,

$$F_Y(0) = P[X^2 \leq 0] = P[X = 0] = 0$$

$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}), & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

②  $\frac{d}{dy}$

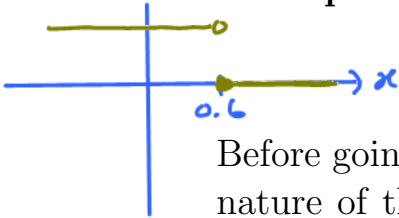
$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(At the boundary ( $y=0$ ), if  $Y$  is a continuous RV,  $f_Y(y)$  can be any value.)

**10.81.** There is no guarantee that a function of continuous random variable will also give a continuous random variable.

**Example 10.82.** Let  $X \sim \mathcal{U}(0, 1)$  and  $Y = g(X)$  where

$$g(x) = \begin{cases} 1, & x < 0.6 \\ 0, & x \geq 0.6. \end{cases}$$

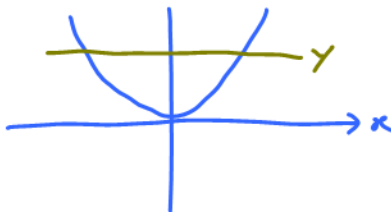


Before going deeply into the math, it is helpful to think about the nature of the derived random variable  $Y$ . The definition of  $g(x)$  tells us that  $Y$  has only two possible values,  $Y = 0$  and  $Y = 1$ . Thus,  $Y$  is a discrete random variable.

**10.83.** Suppose  $X$  is a continuous random variable. To check whether a random variable  $Y = g(X)$  is a continuous random variable, there are two important techniques:

- (a) Check that the cdf  $F_Y(y)$  is a continuous (no jump) function (for all  $y$ ).
- (b) Check that  $P[Y = y] = 0$  for all  $y$ .

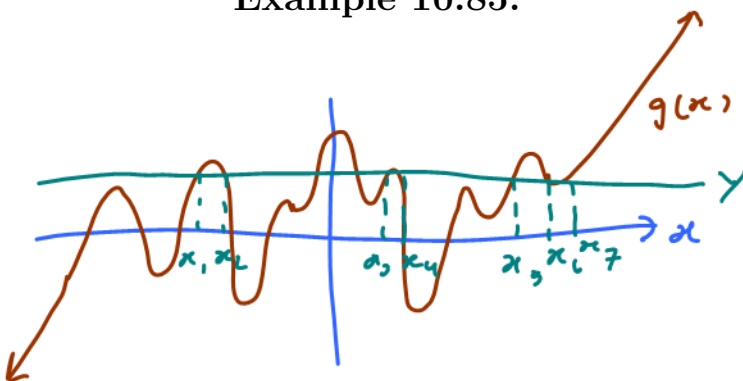
**Example 10.84.** Consider  $Y = X^2$  when  $X$  is a continuous random variable.



$$P[Y=y] = P[X^2=y] = \begin{cases} P[X=\sqrt{y}] + P[X=-\sqrt{y}], & y > 0, \\ P[X=0], & y = 0, \\ 0, & y < 0, \end{cases}$$

$\equiv 0$  for all  $y \Rightarrow Y$  is also a continuous R.V.

**Example 10.85.**



Suppose  $Y = g(x)$  when  $x$  is a

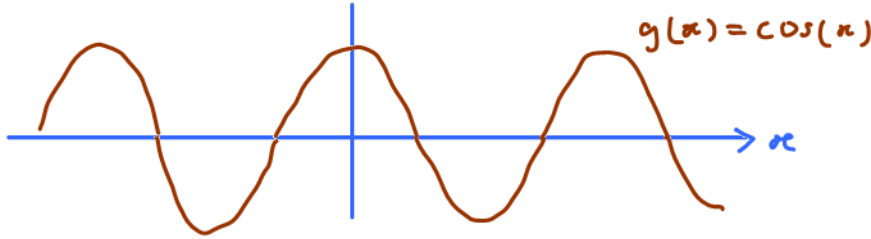
continuous R.V.

$$P[Y=y] = P[X=x_1] + P[X=x_2] + \dots + P[X=x_7]$$

$= 0$

The number of solutions for  $y = g(x)$  is always finite.

**Example 10.86.** Consider  $Y = \cos(X)$  when  $X$  is a continuous random variable.



$P[Y=y] = 0$   
 $\Rightarrow Y$  is a cont. RV.

$$P[Y=0] = P[\cos(X)=0] = P\left[X = \frac{\pi}{2} + k\pi, k=0, \pm 1, \pm 2, \dots\right] = \sum_{k=0, \pm 1, \pm 2, \dots} P\left[X = \frac{\pi}{2} + k\pi\right] = 0$$

**10.87.** Let  $X$  be a continuous random variable and  $Y = g(X)$ . Suppose, for each  $y$ , the collection of  $x$  values that satisfy  $y = g(x)$  is (at most) countable. Then  $Y$  is a continuous random variable.

To see this, let  $\mathcal{B}_y$  be the collection of  $x$  values that satisfy  $y = g(x)$ . Then,  $[Y = y] = [X \in \mathcal{B}_y]$  and hence  $P[Y = y] = P[X \in \mathcal{B}_y]$ . If  $\mathcal{B}_y$  is countable, then we can write  $\mathcal{B}_y$  as a countable disjoint union of events  $[X = x]$  for the  $x \in \mathcal{B}_y$ . By the countable additivity axiom (P3),

$$P[Y = y] = \sum_{x \in \mathcal{B}_y} P[X = x]. \tag{25}$$

Because  $X$  is continuous, we know that  $P[X = x] = 0$  for any  $x$ . Hence, the sum above is 0.

**Example 10.88.** In Example 10.82, there are uncountably many  $x$  values that satisfy  $g(x) = 1$ . Therefore, the condition in 10.87 is not satisfied and hence we can't use 10.87.

**Example 10.89.** Suppose  $X \sim \mathcal{E}(\lambda)$ . Let  $Y = \frac{1}{X^2}$ .

(a) Check that  $Y$  is a continuous random variable.

For  $y > 0$ , the  $x$  values that satisfy  $y = g(x)$  is 1.  
 For  $y \leq 0$ , " " " is 0.  
 $\Rightarrow Y$  is a continuous RV.

(b) Find  $F_Y(y)$ .

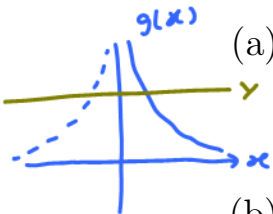
(i) Note that  $\frac{1}{x^2} > 0 \Rightarrow F_Y(y) = 0$  for  $y \leq 0$ .

(ii) For  $y > 0$ ,

$$F_Y(y) = P[Y \leq y] = P\left[\frac{1}{X^2} \leq y\right] = P\left[\frac{1}{y} \leq X^2\right] = P\left[X^2 \geq \frac{1}{y}\right] = P\left[X^2 > \frac{1}{y}\right] + P\left[X^2 = \frac{1}{y}\right] = 1 - P\left[X^2 \leq \frac{1}{y}\right] = 1 - P\left[X \leq \frac{1}{\sqrt{y}}\right]$$

$$F_Y(y) = \begin{cases} 1 - F_X\left(\frac{1}{\sqrt{y}}\right), & y > 0, \\ 0, & \text{otherwise.} \end{cases} = \dots$$

For  $X \sim \mathcal{E}(\lambda)$ ,  $X > 0$ .



167 (x is cont.) plug-in the cdf of X = ...

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\frac{d}{dy} \frac{1}{y} = \frac{d}{dy} y^{-1} = -1 y^{-2} = -\frac{1}{y^2}$$

(c) Find  $f_Y(y)$ .

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} +\frac{1}{2} y^{-\frac{3}{2}} f_x\left(\frac{1}{\sqrt{y}}\right), & y > 0 \\ 0, & y \leq 0 \end{cases} = \begin{cases} \lambda e^{-\frac{\lambda}{\sqrt{y}}} \frac{1}{2} y^{-\frac{3}{2}}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

the  $y=0$  case is added here easily from the shown fact that  $Y$  is a cont. RV.

**Exercise 10.90** (F2011). Suppose  $X$  is uniformly distributed on the interval  $(1, 2)$ . ( $X \sim \mathcal{U}(1, 2)$ .) Let  $Y = \frac{1}{X^2}$ .

- (a) Find  $f_Y(y)$ .
- (b) Find  $\mathbb{E}Y$ .

**Exercise 10.91** (F2011). Consider the function

$$g(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Suppose  $Y = g(X)$ , where  $X \sim \mathcal{U}(-2, 2)$ .

Remark: The function  $g$  operates like a **full-wave rectifier** in that if a positive input voltage  $X$  is applied, the output is  $Y = X$ , while if a negative input voltage  $X$  is applied, the output is  $Y = -X$ .

- (a) Find  $\mathbb{E}Y$ .
- (b) Plot the cdf of  $Y$ .
- (c) Find the pdf of  $Y$ .

|                                    | Discrete                                                                                                                                         | Continuous                                                                                                                                                                   |
|------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $P[X \in B] =$                     | $\sum_{x \in B} p_X(x)$                                                                                                                          | $\int_B f_X(x) dx$                                                                                                                                                           |
| $P[X = x] =$                       | $p_X(x) = F_X(x) - F_X(x^-)$                                                                                                                     | 0                                                                                                                                                                            |
| Interval prob.                     | $P^X((a, b]) = F_X(b) - F_X(a)$<br>$P^X([a, b]) = F_X(b) - F_X(a^-)$<br>$P^X([a, b)) = F_X(b^-) - F_X(a^-)$<br>$P^X((a, b)) = F_X(b^-) - F_X(a)$ | $P^X((a, b]) = P^X([a, b])$<br>$= P^X([a, b]) = P^X((a, b))$<br>$= \int_a^b f_X(x) dx = F(b) - F(a)$                                                                         |
| $\mathbb{E}X =$                    | $\sum_x x p_X(x)$                                                                                                                                | $\int_{-\infty}^{+\infty} x f_X(x) dx$                                                                                                                                       |
| For $Y = g(X)$ ,                   | $p_Y(y) = \sum_{x: g(x)=y} p_X(x)$                                                                                                               | $f_Y(y) = \frac{d}{dy} P[g(X) \leq y]$ .<br>Alternatively,<br>$f_Y(y) = \sum_k \frac{f_X(x_k)}{ g'(x_k) }$ ,<br>$x_k$ are the real-valued roots of the equation $y = g(x)$ . |
| For $Y = g(X)$ ,<br>$P[Y \in B] =$ | $\sum_{x: g(x) \in B} p_X(x)$                                                                                                                    | $\int_{\{x: g(x) \in B\}} f_X(x) dx$                                                                                                                                         |
| $\mathbb{E}[g(X)] =$               | $\sum_x g(x) p_X(x)$                                                                                                                             | $\int_{-\infty}^{+\infty} g(x) f_X(x) dx$                                                                                                                                    |
| $\mathbb{E}[X^2] =$                | $\sum_x x^2 p_X(x)$                                                                                                                              | $\int_{-\infty}^{+\infty} x^2 f_X(x) dx$                                                                                                                                     |
| $\text{Var } X =$                  | $\sum_x (x - \mathbb{E}X)^2 p_X(x)$                                                                                                              | $\int_{-\infty}^{+\infty} (x - \mathbb{E}X)^2 f_X(x) dx$                                                                                                                     |

Table 7: Important Formulas for Discrete and Continuous Random Variables